XXV. *Methodus Inveniendi Lineas Curvas ex proprietatibus Variationis Curvaturæ. Auctore* Nicolao Landerbeck, *Mathes. Profess. in Acad. Upsaliensi Adjuncto : communicated by* Nevil Maskelyne, *D. D. F. R. S. and Astronomer Royal.*

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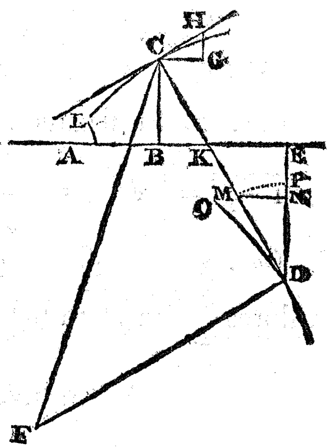
PARS PRIMA.

QUALITAS curvaturæ in diversis lineis diversisque earum punctis diversa reperitur. Circulo ubique eadem est curvatura, quæ in alia quavis curva, continue crescendo vel decrescendo, figuram ab uniformi circuli variat; quo enim majori velocitate progrediens crescit vel decrescit curvaturæ radius, eo citius curvæ a circuli osculatorii curvatura deflectit; et quo majori celeritate isochrona ipsa curva crescit vel decrescit, eo citius fertur motu angulari radius curvedinis et remotius idem curvaturæ gradus locum obtinet, quo circulus curvam osculans eam in angulo majori vel minori in puncto contactus simul secat. Hæc curvaturæ a circulari aberratio, quæ curvaturæ variatio nuncupatur, etsi alia in alia curva gaudeat proprietate, mensurari et exprimi potest generaliter per rationem fluxionum radii curvedinis et curvæ, quæ ratio proinde variationis index censenda est, ut in opere, quod Methodus Fluxionum inscribitur, illustrissimus NEWTONUS nos docuit. Demonstravit præterea MACLAURINUS in propositione trigesima sexta Tractatus de Fluxionibus, quod index hic variationis curvaturæ curvæ cujuscunque sit ut tangens anguli, linea punctum in curva et centrum curvaturæ evolutæ jungente et radio curvaturæ in isto puncto comprehensi; cujus analytica expressione, quæ pro quavis curva calculo differentiali facile habetur, intima curva-

rum examinare licet, ut non ſolum punctum ejuſdem curvæ, ubi inequabilitas curvaturæ eſt vel nulla vel datæ magnitudinis vel minima vel maxima vel infinita determinare, ſed etiam curvas inter ſe comparare valeant matheſeos periti, ut quibus punctis curvatura ſit æqualis et ſimilis diſcernere queant. Methodum ex proprietatibus variationis curvaturæ inveniendi curvas explicatam adhuc non vidi, quæ, ſi detecta et explicata fuerit, quantum matheſeos ſcientiæ interſit, quemque præbeat uſum in problematibus tam mathematicis quam phyſicis ſolvendis, quæ a curvatura dependent, mathematicorum eſt judicare, quorum etiam judicio, quæ ad methodum hanc explicandam feci tentamina ſubjicio.

THEOREMA I.



Si curvæ cujuſdam LC, ad axin concavæ vel convexæ, index variationis curvaturæ, ſeu tangens anguli DCF, radio curvaturæ CD in puncto C et linea CF, punctum C et centrum curvaturæ F evolutæ QD jungente, comprehenſi, dicatur T, ſinus anguli BCD p, poſito ſinu toto 1, arcus curvæ LC z, coordinatæ orthogonales AB, BC x et y earumque fluxiones dp, dz, dx et dy reſpective dicantur, erit $\frac{ddx}{dx} = -\frac{T\,dp}{\sqrt{1-p^2}}$.

Sumatur DM unitati æqualis et ducantur DE axi AB et MN ipſi DE normales, et deſcribatur arcus circuli MP; erit MN $= p$ et DN $= \sqrt{1-p^2}$. Quoniam ob ſimilitudinem triangulorum DNM et CHG, erit DN $(\sqrt{1-p^2})$: MN (p) :: CG (dx) : GH

 (dy)

(dy) et $dy = \frac{p\,dx}{\sqrt{1-p^2}}$, eamque ob cauffam DN $(\sqrt{1-p^2})$: DM (1) :: CG (dx) : CH (dz) et $dz = \frac{dx}{\sqrt{1-p^2}}$. Si radius curvaturæ CD fit R et ponatur conftans, ejus enim fluxio ex coordinatarum non dependet, erit lineæ BE fluxio $= -dx$. Propter fimilitudinem triangulorum CBK, KED et NDM erit DM (1) : MN (p) :: CK + KD (R) : BE $= \mathrm{R}p$, cujus fluxio $\mathrm{R}\,dp = -dx$ et $\mathrm{R} = -\frac{dx}{dp}$, et fi hujus æquationis fluxiones fumantur, pofita dp conftante, habetur $d\mathrm{R} = -\frac{ddx}{dp}$, quæ per $dz = \frac{dx}{\sqrt{1-p^2}}$ divifa dat $\mathrm{T}\left(=\frac{d\mathrm{R}}{dz}\right) = -\frac{ddx\sqrt{1-p^2}}{dx\,dp}$, qua prodit $\frac{ddx}{dx} = -\frac{\mathrm{T}\,dp}{\sqrt{1-p^2}}$.

Cor. 1. Si tangens anguli BCD defignetur per r, erit $p = \frac{r}{\sqrt{1+r^2}}$, $\sqrt{1-p^2} = \frac{1}{\sqrt{1+r^2}}$ et $dp = \frac{dr}{\overline{1+r^2}|^{\frac{3}{2}}}$, unde $\frac{ddx}{dx} = -\frac{\mathrm{T}\,dr}{1+r^2}$.

Cor. 2. Si fecans anguli BCD dicatur S, erit $p = \frac{\sqrt{s^2-1}}{s}$, $\sqrt{1-p^2} = \frac{1}{s}$ et $dp = \frac{ds}{s^2\sqrt{s^2-1}}$, quo $\frac{ddx}{dx} = -\frac{\mathrm{T}\,ds}{s\sqrt{s^2-1}}$.

Cor. 3. Si cofinus q, cotangens t et cofecans v dicantur, valores $\frac{ddx}{dx}$ eandem habent formam, fignis mutatis.

Schol. 1. Quum inventa fit $\mathrm{T} = -\frac{ddx\sqrt{1-p^2}}{dx\,dp}$, methodum habemus perfacilem calculandi generaliter variationem curvaturæ uniufcujufque curvæ; data enim relatione inter fluxiones coordinatarum, quæ per æquationem hujus formæ $dy = \mathrm{X}dx$ exhibetur, ubi X functio eft abfciffæ x, datur $\frac{p}{\sqrt{1-p^2}} = \mathrm{X}$, qua x per p et p per x exprimi poteft. Si variatio curvaturæ per p expreffa defideretur, ponatur $x = \mathrm{P}$, quantitatis p functioni, et fluxioni-

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bus primis $dx = \overset{\prime}{\mathrm{P}}dp$ et fecundis $ddx = \overset{\prime\prime}{\mathrm{P}}dp^2$, pofita dp conftante, fumtis, valoribufque pro dx et ddx fubftitutis, habetur curvæ propofitæ index variationis curvaturæ $\mathrm{T} = -\frac{\overset{\prime\prime}{\mathrm{P}}\sqrt{1-p^2}}{\overset{\prime}{\mathrm{P}}}$, denotantibus $\overset{\prime\prime}{\mathrm{P}}$ et $\overset{\prime}{\mathrm{P}}$ functiones quantitatis p. Si vero index variationis curvaturæ exprimenda fit per x, æquatione $\mathrm{X} = \frac{p}{\sqrt{1-p^2}}$ inveniatur $p = \overset{\prime}{\mathrm{X}}$ et $\sqrt{1-p^2} = \sqrt{1-\overset{\prime}{\mathrm{X}}^2}$, fumtifque æquationis $p = \overset{\prime}{\mathrm{X}}$ primis et fecundis fluxionibus, dp conftante habita, erit $dp = \overset{\prime\prime}{\mathrm{X}}dx$ et $0 = \overset{\prime\prime}{\mathrm{X}}ddx + \overset{\prime\prime\prime}{\mathrm{X}}dx^2$, qua $ddx = -\frac{\overset{\prime\prime\prime}{\mathrm{X}}dx^2}{\overset{\prime\prime}{\mathrm{X}}}$, et fubftitutione debita $\mathrm{T} = \frac{\overset{\prime\prime\prime}{\mathrm{X}}\sqrt{1-\overset{\prime}{\mathrm{X}}^2}}{\overset{\prime\prime}{\mathrm{X}}}$, fignificantibus $\overset{\prime\prime\prime}{\mathrm{X}}$, $\overset{\prime\prime}{\mathrm{X}}$, et $\overset{\prime}{\mathrm{X}}$ functiones abfciffæ x.

Schol. 2. Hoc adhibito theoremate inveniantur curvæ, fi inter T et p, T et r vel T et s detur quædam relatio. Sit enim $\mathrm{T} = \mathrm{P}$, functioni quantitatis p, habetur $\frac{ddx}{dx} = -\frac{\mathrm{P}dp}{\sqrt{1-p^2}}$, et facta integratione $\log. dx = -\int\frac{\mathrm{P}dp}{\sqrt{1-p^2}} + \log. \mathrm{A}dp$, quæ, fi N fit numerus, cujus logarithmus hyperbolicus 1, evadit $\log. dx = -\log. \mathrm{N}\int\frac{\mathrm{P}dp}{\sqrt{1-p^2}} + \log. \mathrm{A}dp$, et fi $\mathrm{N}\int\frac{\mathrm{P}dp}{\sqrt{1-p^2}}$ ponatur F et tranfeundo a logarithmis ad quantitates abfolutas, erit $dx = \frac{\mathrm{A}dp}{\mathrm{F}}$, cujus fi fumantur integralia, obtinetur $x + \mathrm{C} = \int\frac{\mathrm{A}dp}{\mathrm{F}}$, qua equatione p per x exprimi poffit. Sit $p = \mathrm{X}$, functioni abfciffæ x, erit $\sqrt{1-p^2} = \sqrt{1-\mathrm{X}^2}$, $dy\left(= \frac{pdx}{\sqrt{1-p^2}}\right) = \frac{\mathrm{X}dx}{\sqrt{1-\mathrm{X}^2}}$ et integratione $y = \int\frac{\mathrm{X}dx}{\sqrt{1-\mathrm{X}^2}}$ æquatio, qua curvarum natura innotefcit.

Patet

Patet hinc, quod, quoties $\int \frac{Pdp}{\sqrt{1-p^2}}$ per logarithmos fumi non poffit, curva, quæ quæritur, fit tranfcendens; ut vero fit algebraica, requiritur, non folum ut $\int \frac{Pdp}{\sqrt{1-p^2}}$ fit integrale logarithmicum, fed etiam ut $\int \frac{Adp}{F}$ et $\int \frac{Xdx}{\sqrt{1-X}}$ fint quantitates, quæ abfolutam admittant æquationem.

Exempl. 1. Si invenienda fit curva, cujus variatio curvaturæ $T = \frac{3\sqrt{1-p^2}}{p}$. Per theorema habetur $\frac{ddx}{dx}\left(= -\frac{Tdp}{\sqrt{1-p^2}}\right) = -\frac{3dp}{p}$, quam æquationem integrando et corrigendo prodit log. dx $\left(= \text{log. } \frac{1}{p^3} + \text{log.} -\frac{adp}{2}\right) = \text{log. } -\frac{adp}{2p^3}$, et a logarithmis ad quantitates abfolutas tranfeundo $dx = -\frac{adp}{2p^3}$, et iterum integrando et corrigendo $x + C$ $\left(= -\int \frac{adp}{2p^3}\right) = \frac{a}{4p^2}$, ex qua æquatione habetur $p = \frac{\sqrt{a}}{2\sqrt{C+x}}$ et $\sqrt{1-p^2} = \frac{\sqrt{4C+4x-a}}{2\sqrt{C+x}}$, unde fequitur, quod fit y $\left(\left(= \int \frac{pdx}{\sqrt{1-p}}\right) = \int \frac{\sqrt{a}\,.\,dx}{\sqrt{4C+4x-a}}\right) = \sqrt{a}\,.\,\sqrt{4C+4x-a}$, qua æquatione conftat, curvam effe parabolam apollonianam, cujus parameter principalis a.

Exempl. 2. Si curva quæritur, cujus variatio curvaturæ $T = \frac{1-3p^2}{p\sqrt{1-p^2}}$, theoremate habetur $\frac{ddx}{dx}\left(= -\frac{Tdp}{\sqrt{1-p^2}}\right) = \frac{\overline{3p^2-1}\,.\,dp}{p\,.\,\overline{1-p^2}}$, cujus æquatio integralis correcta erit log. dx $\left(= \text{log. } \frac{1}{p\,.\,\overline{1-p^2}} + \text{log. } adp\right) = \text{log. } \frac{adp}{p\,.\,\overline{1-p^2}}$, vel, facto a logarithmis tranfitu, $\frac{dx}{a} = \frac{dp}{p\,.\,\overline{1-p^2}}$ et integratione $\frac{x}{a} + C = \text{log. } \frac{p}{\sqrt{1-p^2}}$, unde fi N fit numerus,

merus, cujus logarithmus hyperbolicus 1, erit $\frac{p}{\sqrt{1-p^2}} = N^{\frac{x}{a}+C}$ et y $(= \int \frac{pdx}{\sqrt{1-p^2}}$ $\int N^{\frac{x}{a}+C} dx$, curva igitur eſt logarithmica.

Exempl. 3. Si curvaturæ variatio ſit $T = \frac{3 \cdot \overline{a^2 \mp b^2} \cdot r}{a^2r^2 \pm b^2}$, quæritur curva. Per corollarium primum habetur $\frac{ddx}{dx}$ $(= -\frac{Tdr}{1+r^2}$ $= -\frac{3 \cdot \overline{a^2 \mp b^2} \cdot rdr}{\overline{a^2r^2 \pm b^2} \cdot \overline{1+r^2}}$ et integratione facta log. dx $(=$ log. $\frac{\overline{1+r^2}|^{\frac{3}{2}}}{\overline{a^2r^2 \pm b^2}|^{\frac{3}{2}}}$ $+$ log. $\pm \frac{b^2a^2dr}{2 \cdot \overline{1+r^2}|^{\frac{3}{2}}}) =$ log. $\pm \frac{b^2a^2dr}{2 \cdot \overline{a^2r^2 \pm b^2}|^{\frac{3}{2}}}$, vel, ſumendo quantitates abſolutas, $\mp dx = \frac{b^2a^2dr}{2 \cdot \overline{a^2r^2 \pm b^2}|^{\frac{3}{2}}}$, et integratione $C \mp x = \frac{a^2r}{2\sqrt{a^2r^2 \pm b^2}}$, ex qua æquatione $r = \frac{b \cdot \overline{2C \mp 2x}}{a\sqrt{\overline{2C \mp 2x}|^2 - a^2}}$ et y $(= \int rdx) =$ $\int \frac{b \cdot \overline{2C \pm 2x} \cdot dx}{a\sqrt{\overline{2C \mp 2x}|^2 - a^2}}$, æquatio indolem curvarum exprimens, quæ ſi $C = \frac{a}{2}$ erit $y = \frac{b\sqrt{ax \mp x^2}}{a}$, æquatio pro ſectionibus conicis.

Exempl. 4. Proponatur invenire curvam, cujus curvaturæ variatio $T = \frac{2 \cdot \overline{2s^2 - 3}}{\overline{s^2 - 2} \cdot \sqrt{s^2 - 1}}$, per ſecantem anguli BCD expreſſa, datur. Per corollarium ſecundum curvam conſequi licet; ſed per ſubſtitutionem $T = \frac{2 \cdot \overline{3p^2 - 1}\sqrt{1-p^2}}{p \cdot \overline{2p^2 - 1}}$ habetur, erit $\frac{ddx}{dx}$ $(= -\frac{Tdp}{\sqrt{1-p^2}}$ $= \frac{2 \cdot \overline{1 - 3p^2} \cdot dp}{p \cdot \overline{2p^2 - 1}}$, integratione log. dx $(=$ log. $\frac{1}{p^2\sqrt{2p^2-1}}$ $+$ log. $adp) =$ log $\frac{adp}{p^2\sqrt{2p^2-1}}$ et adhibendo quantitates ab-

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solutas $dx = \frac{a\,dp}{p^2\sqrt{2p^2 - 1}}$ cujus æquatio integralis $x + C = \frac{a\sqrt{2p^2 - 1}}{p}$ dat $p = \frac{a}{\sqrt{2a^2 - \overline{x + C}|^2}}$ et $\sqrt{1 - p^2} = \frac{\sqrt{a^2 - \overline{x + C}|^2}}{\sqrt{2a^2 - \overline{x + C}|^2}}$, quo $y \left(= \int \frac{p\,dx}{\sqrt{1 - p^2}} \right) = \int \frac{a\,dx}{\sqrt{a^2 - \overline{x + C}|^2}}$ æquatio pro curva, quæ ſinuum vocatur.

THEOREMA II.

Si coſinus anguli BCD ſit q, poſito radio 1, et reliquæ determinationes maneant ut in theoremate præcedenti, erit $\frac{ddy}{dy} = \frac{T\,dq}{\sqrt{1 - q^2}}$.

Nam propter triangulorum DMN et CHG ſimilitudinem MN $(\sqrt{1 - q^2})$: DN (q) :: HG (dy) : CG (dx) et MN $(\sqrt{1 - q^2})$: MD (1) :: HG (dy) : CH (dz) erit $dx = \frac{q\,dy}{\sqrt{1 - q^2}}$ et $dz = \frac{dy}{\sqrt{1 - q^2}}$. Per ſimilitudinem triangulorum CDK, KED, et NDM, erit MD (1) : DN (q) :: DK + KC (R) : y + DE, unde R$q = y$ + DE, ſumptiſque fluxionibus R$dq = dy$, qua R $= \frac{dy}{dq}$, radius enim curvaturæ ut conſtans ſuppoſitus, DE etiam conſtans erit, et ſi ulterius ſumantur fluxiones, dq conſtante habita, erit dR $= \frac{ddy}{dq}$, qua diviſa per $dz = \frac{dy}{\sqrt{1 - q^2}}$ provenit T $\left(= \frac{dR}{dz} \right) = \frac{ddy\sqrt{1 - q^2}}{dy\,dq}$ et $\frac{ddy}{dy} = \frac{T\,dq}{\sqrt{1 - q^2}}$.

Cor. 1. Si cotangens anguli BCD dicatur t, erit $q = \frac{t}{\sqrt{1 + t^2}}$, $\sqrt{1 - q^2} = \frac{1}{\sqrt{1 + t^2}}$, $dq = \frac{dt}{\overline{1 + t^2}|^{\frac{3}{2}}}$ et $\frac{ddy}{dy} = \frac{T\,dt}{1 + t^2}$.

Cor.

Cor. 2. Si cosecans anguli BCD sit v, erit $q = \frac{\sqrt{v^2 - 1}}{v}$, $\sqrt{1 - q^2} = \frac{1}{v}$, $dq = \frac{dv}{v^2 \sqrt{v^2 - 1}}$ et $\frac{ddy}{dy} = \frac{Tdv}{v \sqrt{v^2 - 1}}$.

Schol. 1. Si per æquationem hujus formæ $dx = Ydy$, ubi Y functio est ordinatæ y, relatio datur inter coordinatarum fluxiones æquatione $T = \frac{ddy\sqrt{1 - q^2}}{dydq}$, eodem calculandi modo ac in scholio 1. variatio curvaturæ $T = \frac{\overset{\shortparallel}{Q}\sqrt{1 - q^2}}{\overset{\prime}{Q}}$ generaliter in q habetur, significantibus $\overset{\shortparallel}{Q}$ et $\overset{\prime}{Q}$ functiones cosinus q. Pari calculandi ratione ac in eodem Scholio curvaturæ variatio $T = -\frac{\overset{\prime\prime\prime}{Y}\sqrt{1 - \overset{\prime}{Y}^2}}{\overset{\prime\prime}{Y}}$, denotantibus $\overset{\prime\prime\prime}{Y}$, $\overset{\prime\prime}{Y}$ et $\overset{\prime}{Y}$ functiones ordinatæ y, inveniri potest.

Schol. 2. Per hoc theorema natura curvæ habetur ex data relatione inter T et q, T et r vel T et s, &c. Nam si sit $T = Q$, functioni cosinus q, erit $\frac{ddy}{dy} = \frac{Qdq}{\sqrt{1 - q^2}}$, et integratione log. $dy = \int \frac{Qdq}{\sqrt{1 - q^2}} +$ log. Bdq, vel log. $dy =$ log. $N \int \frac{Qdq}{\sqrt{1 - q^2}} +$ log. Bdq, si N sit numerus, cujus logarithmus hyperbolicus 1; et si $N \int \frac{Qdq}{\sqrt{1 - q^2}}$ dicatur G, et facto a logarithmis transitu, prodit $dy = \frac{Bdq}{G}$, et per integrationem $y + C = \int \frac{Bdq}{G}$ ex qua q in y datur. Sit $q = Y$, functioni ordinatæ y, erit $\sqrt{1 - q^2} = \sqrt{1 - Y^2}$ et $x \left(= \int \frac{qdy}{\sqrt{1 - q^2}}\right) = \int \frac{Ydy}{\sqrt{1 - Y^2}}$ generalis æquatio, indolem curvarum exprimens.

Ad hæc idem eſt obſervandum ac in theoremate præcedenti, quod ſi $\int \frac{Qdq}{\sqrt{1-q^2}}$ integrale ſit logarithmicum et $\int \frac{Bdp}{G}$ et $\int \frac{Ydy}{\sqrt{1-Y^2}}$ quantitates perfecte integrabiles, curva evadit algebraica, ſi vero aliter evenerit, ſemper tranſcendens.

Ex. 1. Propoſitum eſto invenire curvam, cujus variatio curvaturæ $T = \frac{1}{q\sqrt{1-q^2}}$. Per theorema habetur $\frac{ddy}{dy} \left(= \frac{Tdq}{\sqrt{1-q^2}}\right) = \frac{dq}{q \cdot \overline{1-q^2}}$, integratione et correctione peracta, log. dy $\left(= \log. \frac{q}{\sqrt{1-q^2}} + \log. - adq\right) = \log. - \frac{aqdq}{\sqrt{1-q^2}}$, et adhibendo quantitates abſolutas $dy = -\frac{aqdq}{\sqrt{1-q^2}}$, et denuo integrando erit $y + C\left(= -a\int \frac{qdq}{\sqrt{1-q^2}}\right) = a\sqrt{1-q^2}$, unde $\sqrt{1-q^2} = \frac{Y+C}{a}$ et $q = \frac{\sqrt{a^2 - \overline{Y+C}|^2}}{a}$ et $x \left(= \int \frac{qdy}{\sqrt{1-q^2}}\right) = \frac{dy\sqrt{a^2 - \overline{y+C}|^2}}{y+C}$ et ſi $C = 0$ pro venit $x = \int \frac{dy\sqrt{a^2-y^2}}{y}$, qua conſtat, curvam eſſe tractoriam.

Ex. 2. Quænam eſt curva, cujus curvaturæ variatio $T = \frac{3q^2-2}{q\sqrt{1-q^2}}$? Vi theorematis habetur $\frac{ddy}{dy} \left(= \frac{Tdq}{\sqrt{1-q^2}}\right) = \frac{\overline{3q^2-2} \cdot dq}{q \cdot \overline{1-q^2}}$, integratione et correctione log. dy $\left(= \log. \frac{1}{q\sqrt{1-q^2}} + \log. - adq\right) = \log. - \frac{adq}{q\sqrt{1-q^2}}$, hoc eſt $dy = -\frac{adq}{q\sqrt{1-q^2}}$, etiterum integrando $y + C \left(= -a\int \frac{dq}{q\sqrt{1-q^2}}\right) = \frac{a\sqrt{1-q^2}}{q}$, qua habetur $\frac{q}{\sqrt{1-q^2}} = \frac{a}{y+C}$ et $x \left(= \int \frac{qdy}{\sqrt{1-q^2}}\right) = \int \frac{ady}{y+C}$, et ſi $C = 0$, $x = \int \frac{dy}{y}$ æquatio pro logarithmica ordinaria.

THEOREM

THEOREMA III.

Manentibus iisdem ac in theoremate primo, erit $\frac{ddz}{dz} = -\frac{\mathrm{T}dp}{\sqrt{1-p^2}}$ vel etiam $\frac{ddz}{dz} = \frac{\mathrm{T}dq}{\sqrt{1-q^2}}$.

Est enim $dz = \frac{dx}{\sqrt{1-p^2}}$ et $dx = dz\sqrt{1-p^2}$, quare $\mathrm{R}\left(= -\frac{dx}{dp}\right) = -\frac{dz\sqrt{1-p^2}}{dp}$, cujus fluxiones $d\mathrm{R} = -\frac{ddz\sqrt{1-p^2}}{dp}$, posita arcus MP fluxione $\frac{dp}{\sqrt{1-p^2}}$ constante, per dz divisæ dant $\mathrm{T}\left(=\frac{d\mathrm{R}}{dz}\right) = -\frac{ddz\sqrt{1-p^2}}{dzdp}$, qua sequitur $\frac{ddz}{dz} = -\frac{\mathrm{T}dp}{\sqrt{1-p^2}}$. Et quum fluxio arcus circuli æqualis sit negativæ fluxioni complimenti, erit etiam $\frac{ddz}{dz} = \frac{\mathrm{T}dq}{\sqrt{1-q^2}}$.

Cor. Si sint ut antea tangens anguli BCD, r et secans s, habetur $\frac{ddz}{dz} = -\frac{dr}{1+r^2} = -\frac{ds}{s\sqrt{s^2-1}}$.

Schol. 1. Si alterutra æquationum formæ $dx = \mathrm{Z}dz$ et $dy = \mathrm{Z}dz$, inter fluxiones abscissæ vel ordinatæ et curvæ, relatio detur, per formulam $\mathrm{T} = -\frac{ddz\sqrt{1-p^2}}{dzdp}$ vel $\mathrm{T} = \frac{ddz\sqrt{1-q^2}}{dzdq}$, variatio curvaturæ in p, $-\frac{\ddot{\mathrm{P}}\sqrt{1-p^2}}{\dot{\mathrm{P}}}$, in q $\frac{\ddot{\mathrm{Q}}\sqrt{1-q^2}}{\dot{\mathrm{Q}}}$, et in z $\frac{\dot{\mathrm{Z}}\sqrt{1-\mathrm{Z}^2}}{\ddot{\mathrm{Z}}}$, eodem ac antea habetur, posita fluxione quantitatis $\int\frac{dp}{\sqrt{1-p^2}}$ constante.

Schol. 2. Ope hujus theorematis invenire licet indolem curvæ, si inter T et p, T et q, &c. relatio detur. Sit $\mathrm{T} = \mathrm{P}$, functioni

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ſinus p, erit $\frac{ddz}{dz} = -\frac{Pdp}{\sqrt{1-p^2}}$, facta integratione et correctione debita, log. $dz = -\int\frac{Pdp}{\sqrt{1-p^2}} + \log.\frac{Edp}{\sqrt{1-p^2}}$, vel log. $dz = -\log.\, N\int\frac{Pdp}{\sqrt{1-p^2}} + \log\frac{Edp}{\sqrt{1-p^2}}$, ſi N ſit baſis logarithmorum hyperbolicorum, atque poſita $N\int\frac{Pdp}{\sqrt{1-p^2}} = H$, et facto de logarithmis tranſitu, $dz = \frac{Edp}{H\sqrt{1-p^2}}$, et iterum integrando $z + C = \int\frac{Edp}{H\sqrt{1-p^2}}$, unde p per z habetur. Sit $p = Z$, functioni arcus curvæ z, erit $\sqrt{1-p^2} = \sqrt{1-Z^2}$, $x\ (=\int dz\sqrt{1-p^2}) = \int dz\sqrt{1-z^2}$ et $y\ (=\int pdz) = \int Zdz$, quorum alterutra curvarum indoles cognoſcitur. Pari modo procedendum eſt, ſi $T = Q$, quantitatas q functioni.

Hinc facile colligitur, quod, quoties $\int\frac{Pdp}{\sqrt{1-p^2}}$ ſit integrale logarithmicum et quantitates $\int\frac{Edp}{H\sqrt{1-p^2}}$ et $\int dz\sqrt{1-Z^2}$ vel $\int Zdz$ perfectæ integrabiles, curvæ erunt rectificabiles et algebraicæ, quoties relatio inter x et z vel inter y et z in relationem algebraicam x et y reſolvi poſſit.

Exempl. 1. Si deſideretur curva, cujus curvaturæ variatio $T = \frac{2\sqrt{1-p^2}}{p}$. Per theorema eſt $\frac{ddz}{dz}\left(= -\frac{Tdp}{\sqrt{1-p^2}}\right) = -\frac{2dp}{p}$ et integratione log. $dz\ \left(= \log.\frac{1}{p^2} + \log.\frac{adp}{\sqrt{1-p^2}}\right) = \log.\frac{adp}{p^2\sqrt{1-p^2}}$, qua $dz = \frac{adp}{p^2\sqrt{1-p^2}}$, et denuo integrando $z + C = -\frac{a\sqrt{1-p^2}}{p}$, qua habetur

betur $p = \frac{a}{\sqrt{a^2 + \overline{z+C}|^2}}$, $\sqrt{1-p^2} = \frac{z+C}{\sqrt{a^2+\overline{z+C}|^2}}$ et x ($= \int dz\sqrt{1-p^2}$) $= \frac{\overline{z+C}\,.\,dz}{\sqrt{a^2+\overline{z+C}|^2}}$; si $C=0$, evadit x ($= \int \frac{zdz}{\sqrt{a^2-z^2}}$) $= -a + \sqrt{a^2-z^2}$, curva igitur est catenaria.

Exempl. 2. Sit variatio curvaturæ $T = \frac{\sqrt{1-q^2}}{q}$, quæritur curva. Vi theorematis erit $\frac{ddz}{dz}$ ($= \frac{Tdq}{\sqrt{1-q^2}}$) $= \frac{dq}{q}$ et integratione log. dz ($=$ log. q + log. $\frac{adq}{\sqrt{1-q^2}}$) $=$ log. $\frac{aqdq}{\sqrt{1-q^2}}$, qua $dz = \frac{aqdq}{\sqrt{1-q^2}}$ et rursus integrando $z+C = -a\sqrt{1-q^2}$, unde $q = \frac{\sqrt{a^2-\overline{z+C}|^2}}{a}$, $\sqrt{1-q^2} = \frac{z+C}{a}$ et y ($= \int dz\sqrt{1-q^2}$) $= \int \frac{\overline{z+C}\,.\,dz}{a}$, si $C = -a$ patet curvam esse cycloidem.

THEOREMA IV.

Retentis antea adhibitis denominationibus, erit $\frac{dR}{RT} = -\frac{dp}{\sqrt{1-p^2}}$.

Quoniam DM (1) : CD (R) :: $-\frac{dp}{\sqrt{1-p^2}}$: dz habetur $dz = -\frac{Rdp}{\sqrt{1-p^2}}$, quæ æquatio per T multiplicata dat $Tdz = -\frac{RTdp}{\sqrt{1-p^2}}$, et quum $dR = Tdz$, prodit $\frac{dR}{RT} = -\frac{dp}{\sqrt{1-p^2}}$.

Schol. 1. Hujus theorematis subsidio inveniri potest curvarum indoles, si inter R et T detur quædam relatio. Sit $R = K$, quantitatis T functioni, habetur per hoc theorema $\frac{dK}{KT} = -\frac{dp}{\sqrt{1-p^2}}$,

et

et facta integratione $\int \frac{dK}{KT} + C = -\int \frac{dp}{\sqrt{1-p^2}}$. Quoniam $-\int \frac{dp}{\sqrt{1-p^2}}$ arcus est circuli, cujus sinus $\sqrt{1-p^2}$, si ponatur $\int \frac{dK}{KT} + C = n$ et N numerus, cujus logarithmus hyperbolicus 1, erit $\sqrt{1-p^2} = \frac{N^{n\sqrt{-1}} - N^{-n\sqrt{-1}}}{2\sqrt{-1}}$, functioni quantitatis T, unde per hanc æquationem T in p vel substitutione T in q vel r, &c. exprimi potest. Cognita relatione inter T et p vel T et q, r, &c. relationem inter coordinatas vel inter curvam et abscissam vel ordinatam per theoremata præcedentia inveniendi aditus patet.

Hinc facile colligitur, quod quoties $\int \frac{dK}{KT}$ non sit per arcus circulares integrabilis curva semper sit trrnscendens.

Ex. 1. Quænam est curva, si relatio inter R et T per æquationem $R = \frac{a \cdot \overline{4+T^2}}{4}$ detur. Theorematis auxilio erit $\frac{2dT}{4+T^2} \left(= \frac{dR}{RT}\right) = -\frac{dp}{\sqrt{1-p^2}}$ et integratione $\int \frac{2dT}{4+T^2} + C = -\int \frac{dp}{\sqrt{1-p^2}}$, ubi $\int \frac{2dT}{4+T^2}$ arcus est circuli, cujus sinus $\frac{T}{\sqrt{4+T^2}}$ et $-\int \frac{dp}{\sqrt{1-p^2}}$ arcus, cujus sinus $\sqrt{1-p^2}$, si arcus constantis C sinus sit c, erit $\frac{T\sqrt{1-c^2} + 2C}{\sqrt{4+T^2}} = \sqrt{1-p^2}$, qua æquatione T in p invenire licet. Si $C = 0$, habetur in hoc casu speciali $T = \frac{2\sqrt{1-p^2}}{p}$ et per theorema 1. $dy = \frac{adx}{\sqrt{2ax+x^2}}$, curva igitur quæsita est catenaria.

Ex. 2. Quæritur curva, si $R = \frac{a\sqrt[4]{1+4T^2}}{2}$. Vi theorematis obtinetur $-\frac{2dT}{1+4T^2} \left(= \frac{dR}{RT}\right) = \frac{dq}{\sqrt{1-q^2}}$ et integrando $-\int \frac{2dT}{1+T^2} + C$

=

$= \int \frac{dq}{\sqrt{1-q^2}}$. Itaque quum arcuum $- \int \frac{2dT}{1+4T^2}$ et $\int \frac{dq}{\sqrt{1-q^2}}$ ſinus ſint $\frac{1}{\sqrt{1+4T^2}}$ et q reſpective, ſi arcus conſtantis C ſinus ſit c, prodit $\frac{\sqrt{1-C^2}+2CT^2}{\sqrt{1+4T^2}} = q$, qua T in q habetur. Si C = 0, erit $T = -\frac{\sqrt{1-q^2}}{2q}$ et per theorema 2. prodit $dx = -\frac{y^2 dy}{\sqrt{a^4 - y^4}}$, unde conſtat, quod in hoc caſu curva ſit elaſtica.

THEOREMA V.

Manentibus adhibitis denominationibus et dicta DF, S, erit $\frac{ds}{ST} - \frac{dT}{T^2} = -\frac{dp}{\sqrt{1-p^2}}$.

Quoniam 1 : T :: CD (R) : DF (S), erit S = RT et R = $\frac{S}{T}$ ejuſque fluxiones $dR = \frac{dS}{T} - \frac{SdT}{T^2}$. Quum vero $\frac{dR}{RT} = -\frac{dp}{\sqrt{1-p^2}}$, prodit ſubſtitutione $\frac{ds}{ST} - \frac{dT}{T^2} = -\frac{dp}{\sqrt{1-p^2}}$.

Schol. Mediante hoc theoremate indagantur curvæ, data relatione inter S et T. Si enim ſit S = L, quantitatis T functioni, habetur $\frac{TdL - LdT}{LT^2} = -\frac{dp}{\sqrt{1-p^2}}$ et integratione $\int \frac{TdL-LdT}{LT^2} + C = -\int \frac{dp}{\sqrt{1-p}}$. Ponatur $\int \frac{TdL-LdT}{LT^2} + C = m$ et N baſis logarithmorum hyperbolicorum, erit $\sqrt{1-p^2} = \frac{N^{m\sqrt{-1}} - N^{-m\sqrt{-1}}}{2\sqrt{-1}}$, quæ functio eſt quantitatis T, quare T in p vel ſubſtitutione in q, r, &c. per hanc æquationem exprimi poteſt. Relatione adepta inter T et p vel q, &c. relatio inter coordinatas, vel inter curvam et abſciſſam vel ordinatam habetur, ut antea expoſitum eſt.

4 Generaliter

Generaliter conſtat, quod, quoties $\int \frac{TdL - LdT}{LT^2}$ non ſit per arcus circulares integrabilis, curva ſit tranſcendens.

Ex. 1. Si radius curvaturæ evolutæ $S = \frac{aT \cdot \overline{9+T^2}|^{\frac{3}{2}}}{54}$, quæritur curva. Per theorema obtinetur $\frac{3dT}{9+T^2}$ $(= \frac{dS}{ST} = \frac{dT}{T^2} = -\frac{dp}{\sqrt{1-p^2}}$ et integratione $\int \frac{3dT}{9+T^2} + C = -\int \frac{dp}{\sqrt{1-p^2}}$. Quum vero arcuum $\int \frac{3dt}{9+T^2}$ et $-\int \frac{dp}{\sqrt{1-p^2}}$ ſinus ſint $\frac{T}{\sqrt{9+T^2}}$ et $\sqrt{1-p^2}$, ſi arcus conſtantis C ſinus ſit c, erit $\frac{T\sqrt{1-c^2}+3C}{\sqrt{9+T^2}} = \sqrt{1-p^2}$ et reſoluta hac æquatione T in p habetur. Si ſit $c=0$, erit $T = \frac{3\sqrt{1-p}}{p}$ et per theorema 1. $y = \sqrt{ax}$, curva igitur in hoc caſu eſt parabola Apolloniana.

Ex. 2. Quænam eſt curva, ſi evolutæ curvaturæ radius $s = \frac{aT \cdot \overline{9+4T^2}|^{\frac{3}{4}}}{2\sqrt{27}}$? Theoremate habetur $\frac{6dT}{9+4T^2} = -\frac{dp}{\sqrt{1-p^2}}$ et integratione $\int \frac{6dT}{9+4T^2} + C = -\int \frac{dp}{\sqrt{1-p^2}}$. Arcuum $\int \frac{6dT}{9+4T^2}$ et $-\int \frac{dp}{\sqrt{1-p^2}}$, ſinus ſunt $\frac{2T}{\sqrt{9+4T^2}}$ et $\sqrt{1-p^2}$, ſi arcus conſtantis C ſinus ponatur c, prodit $\frac{2T\sqrt{1-c^2}+3C}{\sqrt{9+4T^2}} = \sqrt{1-p^2}$, per quam T in p obtinetur, quæ, in caſu $c=0$, dat $T = \frac{3\sqrt{1-p^2}}{2p}$ et theoremate 1. $dy = \frac{a^2dx}{\sqrt{x^4-a^4}}$ æquatio ad curvam, quæ conſtruitur rectificatione ellipſeos et hyperbolæ æquilateræ conjunctim.

THEOREMA VI.

Dicatur CF, U et reliquis manentibus, erit $\frac{dU}{UT} - \frac{dT}{1+T^2} = -\frac{dp}{\sqrt{1-p^2}}$.

Quum enim $1 : \sqrt{1-T^2} :: \text{CD (R)} : \text{CF (U)}$, erit $R = \frac{U}{\sqrt{1+T^2}}$ ejusque fluxio $dR = \frac{dU}{\sqrt{1+T^2}} - \frac{UTdT}{\overline{1+T}|^{\frac{3}{2}}}$, et quum $\frac{dR}{RT} = \frac{dp}{\sqrt{1-p^2}}$, provenit substitutione $\frac{dU}{UT} - \frac{dT}{1+T^2} = -\frac{dp}{\sqrt{1-p^2}}$.

Schol. Auxilio hujus theorematis, curvæ inveniuntur, quando inter T et U relatio detur. Nam si sit $U = M$, functioni quantitatis T, erit per hoc theorema $\frac{\overline{1+T^2}\,.\,dM - MTdT}{MT\,.\,\overline{1+T^2}} = -\frac{dp}{\sqrt{1-p^2}}$. et integratione $\int \frac{\overline{1+T^2}\,.\,dM - MTdT}{MT\,.\,\overline{1+T^2}} + C = -\int \frac{dp}{\sqrt{1-p^2}}$. Itaque, posita basi logarithmica N et $\int \frac{\overline{1+T^2}\,.\,dM - MTdT}{MT\,.\,\overline{1+T^2}} + C = k$, erit $\sqrt{1-p^2} = \frac{N^{k\sqrt{-1}} - N^{-k\sqrt{-1}}}{2\sqrt{-1}}$, quantitatis T functioni, quare inter T et p habetur relatio, per quam, methodo antea exposita, relationem inter coordinatas vel curvam et abscissam sive ordinatam invenire licet.

Consequitur hinc, quod, quando $\int \frac{\overline{1+T^2}\,.\,dM - MTdT}{MT\,.\,\overline{1+T^2}}$ per quadraturam circuli non obtinetur, curva semper sit transcendens.

Ex. Si curva quæritur ubi linea CF sive $U = \frac{a}{2}$, theorematis ope erit $-\frac{dT}{1+T^2} = \frac{dq}{\sqrt{1-q^2}}$ et integratione $-\int \frac{dT}{1+T^2} + C = \int \frac{dq}{\sqrt{1-q^2}}$.

Quum arcuum $-\int\frac{dT}{1+T^2}$ et $\int\frac{dq}{\sqrt{1-q^2}}$ ſinus ſint $\frac{1}{\sqrt{1+T^2}}$ et q ſi arcus conſtantis C ſinus ſit c, obtinetur æquatio $\frac{\sqrt{1-c^2}+CT}{\sqrt{1+T^2}}=q$, qua T in q datur, et ſi $c=0$, $T=\frac{\sqrt{1-q^2}}{q}$, quare in hoc caſu ſpeciali per theorema 2. habetur $dx=-\frac{2\sqrt{y}\,.\,dy}{\sqrt{a-2y}}$, æquatio pro cycloide ordinaria cujus circuli generatoris diameter $\frac{a}{4}$.

THEOREMA VII.

Si variatio curvaturæ evolutæ dicatur V ceteris manentibus, erit $\frac{dT}{V-T\,.\,T}=-\frac{dp}{\sqrt{1-p^2}}$.

Quoniam DM (1) : CD (R) :: $-\frac{dp}{\sqrt{1-p^2}}:dz$, habetur $dz=-\frac{Rdp}{\sqrt{1-p^2}}$, quæ ſi multiplicetur per T prodit $dR\ (=Tdz)=-\frac{RTdp}{\sqrt{1-p^2}}$, et propter 1 : T :: CD (R) : DF erit evolutæ radius curvaturæ $DF=RT$, cujus fluxio $RdT+TdR$ per fluxionem evolutæ diviſa dat ejus curvaturæ variationem V $\left(=\frac{RdT}{dR}+T\right)$ $=-\frac{dT\sqrt{1-p}}{Tdp}+T$ atque inde $\frac{dT}{V-T\,.\,T}=-\frac{dp}{\sqrt{1-p^2}}$.

Schol. Hoc mediante theoremate invenire valemus curvas, ſi inter curvaturæ variationes V et T relatio detur. Sit enim $V=H$, functioni quantitatis T, erit vi theorematis $\frac{dT}{H-T\,.\,T}=-\frac{dp}{\sqrt{1-p^2}}$ et integrando $\int\frac{dT}{H-T\,.\,T}+C=-\int\frac{dp}{\sqrt{1-p^2}}$, ſi itaque ponatur $\int\frac{dT}{H-T\,.\,T}+C=l$ et N baſis logarithmica, erit $\sqrt{1-p^2}=$

$\frac{N^{l\sqrt{-1}} - N^{-l\sqrt{-1}}}{2\sqrt{-1}}$, qua æquatione T in p vel ſubſtitutione in q, r, &c. exprimi poteſt, unde via, æquationem ad curvam inveniendi, patet.

Curva ſemper eſt tranſcendens, quoties $\frac{dT}{\overline{H-T}.T}$ per circuli rectificationem non habetur.

Exempl. Sit evolutæ variatio curvaturæ $V = T + \sqrt{T^2 - 4}$, quæritur curva. Theoremate hoc habetur $\frac{dT}{T\sqrt{T^2-4}} \left(= \frac{dT}{\overline{H-T}.T}\right) = \frac{dq}{\sqrt{1-q^2}}$ et integratione $\int \frac{dT}{T\sqrt{T^2-4}} + C = \int \frac{dq}{\sqrt{1-q^2}}$ arcus, quorum ſinus ſunt $\frac{\sqrt{T+\sqrt{T^2-4}}}{\sqrt{2T}}$ c, et q, ſi arcus conſtantis C ſinus ponatur c, et exinde conſequitur $\frac{\sqrt{1-c^2}\sqrt{T+\sqrt{T^2-4}} + c\sqrt{T-\sqrt{T^2-4}}}{\sqrt{2T}} = q$, qua ſi $c = 0$ prodit $T = \frac{1}{q\sqrt{1-q^2}}$ et per theorema 2. $dx = \frac{dy\sqrt{a^2-y^2}}{y}$ in quo caſu curva eſt tractoria.